If the analysts "can not comprehend the infinite" why do they employ the symbol of the infinite so freely in their equations and decide without hesitation so many questions against the Alexandrian geometer? The analysts make large use of the symbol ∞ in their equations. Do they or do they not comprehend the meaning of the symbolism employed? If they find ∞ incomprehensible, can they not obtain all legitimate results by the aid of *finite* quantities alone?

DEVELOPMENT OF SINθ AND COSθ.

By J. M. BANDY, Trinity College, Trinity, North Carolina.

In discussing the power of the calculus with my own students in Trinity College, I, several years ago, sprung the question "why can the trigonometric functions, sine and cosine, be developed by series?"

The calculus very readily furnished the series; but it did not expose the exponential nature of the functions.

The fact that the value of the functions can be expressed by series forced me to the conclusion that the reason existed in the nature of the functions themselves, and, therefore, they should yield this result directly.

Before proceeding to obtain the series directly from the functions, it will be necessary to produce a series involving an exponential function. The object thereafter will be to trace the law which connects sine and cosine with this exponential function.

We will develop $\left.\left(1+\frac{1}{x}\right)^x\right]_\infty$ which gives us a simple converging series. This series can be made to express an exponential function.

Denoting $\left.\left(1+\frac{1}{x}\right)^x\right]_\infty$ by e; that is, as x increases indefinitely, the *limiting value* of this function $\left.\left(1+\frac{1}{x}\right)^x\right]_\infty$ is e.

$\therefore\ e=1+1+\frac{1}{1.2}+\frac{1}{1.2.3}$, etc.* From this we get

$$e^\theta=\left\{\left.\left(1+\frac{1}{x}\right)^x\right]_\infty\right\}^\theta=1+\theta+\frac{\theta^2}{1.2}+\frac{\theta^2}{1.2.3}+\text{etc.},\ \ldots\ldots\ldots\ldots(1),$$

$$e^{\frac{1}{x}}=\left\{\left.\left(1+\frac{1}{x}\right)^x\right]_\infty\right\}^{\frac{1}{x}}=1+\frac{1}{\infty},\ \ldots\ldots\ldots\ldots(2),$$

*This gives $e=2.71828$, the Naperian base.

and $\log\left(1+\frac{1}{\infty}\right)=\frac{1}{\infty}\log e$ ....(3).

To expose the principles which connect $\sin\theta$ and $\cos\theta$ with the above equations, and thus show that they can be expressed by series.

By geometry, $\cos^2\theta+\sin^2\theta=1$ ....(4).

The first member of (4) may be expressed thus: $\cos^2\theta-(-\sin^2\theta)=1$.

(4), therefore, becomes $\cos^2\theta-(-\sin^2\theta)=1$.......................... (5).

Factoring first member of (5), we have,

$$(\cos\theta+\sin\theta\sqrt{-1})(\cos\theta-\sin\theta\sqrt{-1})=1 \qquad (6).$$

Taking log. of (6), we have $\log(\cos\theta+\sin\theta\sqrt{-1})+\log(\cos\theta-\sin\theta\sqrt{-1})=0$,

or $\log(\cos\theta+\sin\theta\sqrt{-1})=-\log(\cos\theta-\sin\theta\sqrt{-1})$..........................(7).

Denoting either member of (7) by y, we have,

$$\left.\begin{array}{r}\log(\cos\theta+\sin\theta\sqrt{-1})=y,\\ \text{and } \log(\cos\theta-\sin\theta\sqrt{-1})=-y,\end{array}\right\} \qquad (8).$$

$\therefore$ $\cos\theta+\sin\theta\sqrt{-1}=10^y$, ...(9), and $\cos\theta-\sin\theta\sqrt{-1}=10^{-y}$.....(10).

Summing (9) and (10), $2\cos\theta=10^y+10^{-y}$..........................(11).

By trigonometry, $\cos^2\frac{1}{2}\theta=\frac{1}{2}(1+\cos\theta)=\frac{1}{4}(2+2\cos\theta)$

$=\frac{1}{4}(10^y+2+10^{-y})$, [from (11)]..............(12),

and $-\sin^2\frac{1}{2}\theta=\frac{1}{2}(\cos\theta-1)=\frac{1}{4}(2\cos\theta-2)=\frac{1}{4}(10^y-2+10^{-y})$, [from (11)]....(13).

Extracting square roots of (12) and (13),

$$\cos\tfrac{1}{2}\theta=10^{\frac{y}{2}}+10^{-\frac{y}{2}}, \qquad (14),$$

$$\text{and } \sin\tfrac{1}{2}\theta\sqrt{-1}=10^{\frac{y}{2}}-10^{-\frac{y}{2}} \qquad (15).$$

Adding (14) and (15), $\cos\frac{1}{2}\theta+\sin\frac{1}{2}\theta\sqrt{-1}=10^{\frac{y}{2}}$.....................(16).

Comparing (16) and (9), we see that θ may be changed into $\frac{1}{2}\theta$, provided that y is changed into $\frac{1}{2}y$. The same changes may, therefore, be made in (16): $\frac{1}{2}\theta$ may be changed into $\frac{1}{4}\theta$, if $\frac{1}{2}y$ is changed into $\frac{1}{4}y$. (16), therefore, becomes

$$\cos\tfrac{1}{4}\theta+\sin\tfrac{1}{4}\theta\sqrt{-1}=10^{\frac{y}{4}} \qquad (17).$$

Repeating this change, we have, $\cos\frac{1}{8}\theta+\sin\frac{1}{8}\theta\sqrt{-1}=10^{\frac{y}{4}}$ ....(18).

Thus we see that θ may be divided by any power of 2, however great, provided y is divided by the same power.

Let, then, $m=2^n$ ....(19).

We then have, $\cos\frac{1}{m}\theta+\sin\frac{1}{m}\theta\sqrt{-1}=10^{\frac{y}{m}}$ ....(20).

Taking log of (20), we have, $\log(\cos\frac{1}{m}\theta+\sin\frac{1}{m}\theta\sqrt{-1})=\frac{y}{m}$ ...... (21).

But when n in (19) becomes infinite, m becomes infinite.

$\therefore$ $\cos\frac{1}{m}\theta$ in the limit equals 1, and $\sin\frac{1}{m}\theta\sqrt{-1}$ in the limit equals the arc. $\therefore$ (21) becomes $\log(1+\frac{\theta}{m}\sqrt{-1})=\frac{y}{m}$ ....(22).

But from (3), (22) becomes $\frac{\theta}{m}\sqrt{-1}\log e=\frac{y}{m}$, or $y=\theta\sqrt{-1}\log e$ .....(23).

Substituting this value of y in (8), $\log(\cos\theta+\sin\theta\sqrt{-1})=\theta\sqrt{-1}\log e$..(24),

and $\log(\cos\theta-\sin\theta\sqrt{-1})=-\theta\sqrt{-1}\log e$..............(25).

Whence $\cos\theta+\sin\theta\sqrt{-1}=e^{\theta\sqrt{-1}}$...............................(26),

and $\cos\theta-\sin\theta\sqrt{-1}=e^{-\theta\sqrt{-1}}$ ....(27).

Adding (26) and (27), and dividing by 2, $\cos\theta=\frac{1}{2}(e^{\theta\sqrt{-1}}+e^{-\theta\sqrt{-1}})$ ....(28),

by subtracting (27) from (26), and multiplying by $\sqrt{-1}$,

$$\sin\theta=-\tfrac{1}{2}(e^{\theta\sqrt{-1}}-e^{-\theta\sqrt{-1}})\sqrt{-1}\ldots\ldots(29).$$

(28) and (29) enable us to develop $\cos\theta$ and $\sin\theta$ in a series arranged according to the powers of θ. Since $(\theta\sqrt{-1})^2=-\theta^2$, $(\theta\sqrt{-1})^3=-\theta^3\sqrt{-1}$, $(\theta\sqrt{-1})^4=\theta^4$, the substitution of $\theta\sqrt{-1}$ for θ in (1), gives

$$e^{\theta\sqrt{-1}}=1+\theta\sqrt{-1}-\frac{\theta^2}{1.2}-\frac{\theta^3\sqrt{-1}}{1.2.3}+\frac{\theta^4}{1.2.3.4}+\frac{\theta^3\sqrt{-1}}{1.2.3.4.5}\ldots\ldots(30),$$

$$\text{and } e^{-\theta\sqrt{-1}}=1-\theta\sqrt{-1}-\frac{\theta^2}{1.2}+\frac{6^3\sqrt{-1}}{1.2.3}+\frac{\theta^4}{1.2.3.4}-\frac{\theta^3\sqrt{-1}}{1.2.3.4.5}\ldots\ldots(31).$$

Half the sum of (30) and (31) by (28) gives

$$\cos\theta=1-\frac{\theta^2}{1.2}+\frac{\theta^4}{1.2.3.4}-\frac{\theta^6}{1.2.3.4.5.6}+\text{ etc.,}$$

and half the difference of (30) and (31) by (29) gives

$$\sin\theta=\theta-\frac{\theta^3}{1.2.3}+\frac{\theta^3}{1.2.3.4.5}-\text{ etc.}$$

The above are the required series. It is hoped that the law connecting $\cos\theta$ and $\sin\theta$ has been made plain.

(28) and (28) are Euler's results reached in a different way.

From (28) and (29) Demoivre's Theorem, which enables us to obtain the n roots of $y^n+1=0$ and $y^n-1=0$, is derived.

November 4, 1893.

ARITHMETIC.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

63. Proposed by J. A. CALDERHEAD, M. Sc., Professor of Mathematics in Curry University, Pittsburg, Pennsylvania.

I owe A \$100 due in 2 years, and \$200 due in 4 years; when will the payment of \$300 equitably discharge the debt, money being worth 6% ?

III. Solution by the PROPOSER.

Let x=equated time.

Now the amount of \$100 for $(x-2)$ years+the present worth of \$200 due $(4-x)$ years hence must=\$300.

$100+6(x-2)$=amount of \$100 for $(x-2)$ years at 6%.

$\frac{10000}{62-3x}$=present worth of \$200 due $(4-x)$ years hence at 6%.

$$\therefore\ 100+6(x-2)+\frac{10000}{62+3x}=300.$$

$\therefore\ x=3.31533+$years=3 years, 3 months, 24 days.

PROOF. \$107.89=amount of \$100 for 1.31533 years at 6%.

\$192.11=present worth of \$200 due 0.68467 year hence at 6%.

\$107.89+\$192.11=\$300.

QUERY: Will the answers prove as obtained to the solutions of this problem on page 238, Vol. III.?